Bending and Vibration Analysis of Curved FG Nanobeams via Nonlocal Timoshenko Model

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Abstract: The bending and vibration behavior of a curved FG nanobeam using the nonlocal Timoshenko beam theory is analyzed in this paper. It is assumed that the material properties vary through the radius direction. The governing equations were obtained using Hamilton principle based on the nonlocal Timoshenko model of curved beam. An analytical approach for a simply supported boundary condition is conducted to analyze the vibration and bending of curved FG nanobeam. In the both mentioned analysis, the effect of significant parameter such as opening angle, the power law index of FGM, nonlocal parameter, aspect ratio and mode number are studied. The accuracy of the solution is examined by comparing the results obtained with the analytical and numerical results published in the literatures.

Keywords: Curved nanobeam; Vibration and bending; Nonlocal elasticity theory; Analytical solution; Functionally graded curved beam.

1. Introduction

Nowadays various fields of science and industry namely electronics, optics, aerospace, mechanical and civil engineering are considerably affected by nanotechnology^[1, 2]. To precisely design these structures having the knowledge of mechanical properties of nanostructure is strongly needed. For this purpose, numerous investigations are conducted to propose theories and methods to study the mechanical properties of such structures. Nonlocal theory that was proposed by Eringen is one of the best and commonest approaches used by researchers' community^[3-21]. Based on this theory the stress at a reference point is a function of the strain at all neighbor points in the body. The natural frequency of nanotubes with consideration of surface effects using the nonlocal Timoshenko beam theory is studied in Lee *et al.* investigation^[22]. In another study Wang *et al.* analyzed the free vibration of micro/nanobeams based on Eringen's nonlocal elasticity theory and Timoshenko beam theory^[23].

Currently nanocomposites are widely utilized in science and industry but because of their distinct interface which is accompanied with sudden variation in material properties they tend to lose their reliability^[10]. To solve this phenomenon, there is a need to introduce a material, which has a smooth variation in its material properties. Functionally Graded Materials (FGM) have this characteristic which has made them to be largely used in various shape in numerous industries^[2, 24, 25]. The free vibration of piezoelectric FG beams with rectangular cross sections in pre/post-buckling regimes is analyzed by Komijani *et al.*^[26] based on Timoshenko beam theory. Elathar *et al.* analyzed the static- buckling behavior of functionally graded nanobeams as a core structure of micro and nano electro mechanical systems. The principle of virtual displacement is deployed to obtain the equilibrium equations. It was concluded that parameters such as material gradient index, boundary conditions and nonlocal effect have considerable influence on the static- buckling behavior of FG nanobeam^[27]. They studied the static and buckling behavior of nonlocal FG Timoshenko nanobeam in their other investigation which showed that the material distribution profile influences the buckling and

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bending behavior of FG nanobeams^[28]. Rahmani *et al.* used nonlocal beam theory to derive a closed-form solution for Timoshenko FG nanobeam to analyze the size effect on vibration behavior of the functionally graded nanobeam^[24]. It was concluded that parameters such as the gradient index, length scale parameter and length-to-thickness ratio have crucial effect in investigation of free vibration of FG nanobeam. Simsek *et al.* examined the static bending and buckling of a FG nanobeam according to the nonlocal Timoshenko and Euler–Bernoulli beam theory. They concluded that their new proposed model produces larger deflection and smaller buckling load than the classical (local) beam model^[29].

Due to the crucial role of curved micro/nanobeam in designing the engineering structures, various investigations are conducted to study the mechanical behavior of these structures^[30]. Farshi *et al.* analyzed the dynamic behavior of curved nanobeam considering the small scale effect. Their results showed a considerable difference between the results derived from their proposed model and those of classical theories^[31]. Wang *et al.* analyzed the free vibration of nanorings/arches using the Eringen nonlocal theory with consideration of the small scale effect^[32]. In another study made by Madina *et al.* the asymmetric buckling of a shallow initially curved microbeam subjected to distributed nonlinear deflection-dependent electrostatic forced was analyzed^[33].

Due to the previous literature review, there is no investigation that analyzed bending and vibration of FG curved nanobeam and almost all of the investigations were about straight nanobeams. Hence, the purpose of this paper is filling this gap. In this paper, the nonlocal Timoshenko beam theory is used to analyze the free vibration and bending of the curved FG nanobeams. Material properties are assumed to change through thickness direction. Using the Timoshenko and nonlocal theory of curved beam, the governing equations were derived. To analyze the vibration and bending of curved FG nanobeam an analytical approach for a simply supported boundary condition is employed. The effects of parameters such as the opening angle, the power law index of FGM, nonlocal parameter, aspect ratio and mode number on the vibration and bending of curved nanobeam is studied. Due to the negligible differences between the results of this paper and those of previous literature, it was concluded that our results are in excellent agreement with previous studies.

2. Functionally Graded Materials properties

It is assumed that the curved nanobeam is a mixture of steel and alumina. According to present method, the gradient variation may be chosen arbitrarily, though to simplify the mathematical treatment, it is common for them to be shown as exponential-type dependence. However, from an experimental point, it seems constructive to employ a power law gradient. Therefore, the effective material properties of the curved nanobeam are obtained by:

$$\begin{aligned} E_f(z) &= (E_c - E_m) \left(\frac{z}{h} + \frac{1}{2} \right)^p + E_m \\ \rho_f(z) &= (\rho_c - \rho_m) \left(\frac{z}{h} + \frac{1}{2} \right)^p + \rho_m \end{aligned} \quad (1)$$

P denotes the non-negative constant which describes the volume fraction. Subscripts b and t represents the bottom and top of the curved nanobeam respectively. According to aforementioned subscripts when

$z = -h/2$, $E = E_b$ and $z = h/2$, $E = E_t$. For the sake of simplicity the bottom and the top surface of the FG curved nanobeam is assumed to be purely steel and purely alumina, respectively. Although such a power-law relation is not approximate, the above power-law relation can be easily dominated using the variation of the volume fraction of steel and alumina from the bottom to the top in experimental tasks. The variation profile of material properties across the thickness of the beam is denoted by P . Based on distribution function, when $P = 0$, the material is homogenous.

3. Nonlocal Theory

Based on Eringen nonlocal theory, the stress field at point X does not only depend to the strain of the same point and is also depend to strain of all other points of the body. The aforementioned fact is proven by the atomic theory of lattice dynamic and experimental observation of phonon dispersion. The stress tensor σ at point X is calculated as follows:

$$\bar{\sigma} = \int_{\Omega} K(|X' - X, \tau|) \sigma(X') dX' \quad (2)$$

σ represents the classical, microscopic second Piola-Kirchhoff stress tensor at point X, the kernel function $K(|X' - X, \tau|)$ denotes the nonlocal modulus, $(X' - X)$ is the distance and τ stands for material parameter which depends on internal and external characteristic length. Based on the generalized Hooke's law the macroscopic stress σ at point X in Hookean solid is related to the strain at point X which is as follows:

$$\sigma(X) = C(X) : \varepsilon(X) \quad (3)$$

The fourth-order elasticity tensor which denotes double-dot product is represented by C . Eqs. (2) and (3) together represent the nonlocal constitutive behavior of Hookean solid. The weighted average of the contributions of the strain field of all points in the body to the stress field at point X is indicated by Eq. (2). For the sake of simplicity, an equivalent differential model is used instead of integral constitutive relation, which is evaluated as follows:

$$(1 - \mu \nabla^2) \bar{\sigma} = \sigma, \quad \mu = \tau^2 \ell^2 = e_0^2 a^2 \quad (4)$$

τ is determined by $\tau = e_0 a / \ell$ where e_0 is a constant which varies based on each material and a and ℓ represents the internal and external characteristic length. The nonlocal parameter which is represented by μ varies in accordance with different materials. For an elastic material in the one dimensional case, the nonlocal constitutive relations may be simplified as:

$$\sigma(x) - (e_0 a)^2 \frac{\partial^2 \sigma(x)}{\partial x^2} = E \varepsilon(x) \quad (5)$$

E is the Young's modulus. For Timoshenko nonlocal FG beam, Eq. (5) can be written as:

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E(z) \varepsilon_{xx} \quad (6)$$

4. Governing Equations

Using the Hamilton principle and energy approach the governing equations are obtained in this section. For this purpose assume a curved beam with radius R and angle α which is shown in Fig. 1. According to the Timoshenko beam theory the displacement fields for curved beam are as follows:

Figure 1. Schematic of functionally graded curved beam

$$\begin{aligned} u_s(s, z, t) &= -\left(1 + \frac{z}{R}\right) u(s, t) - z \varphi(s, t) \\ u_z(s, z, t) &= -w(s, t) \end{aligned} \quad (7)$$

Where u and w denotes the radial and tangential displacements at any points of the curved beam, respectively. u_0 and w_0 represents the displacement at the centerline and ψ shows the torsion at the cross section. The strains and curvatures at the centerline of a curved nanobeam are calculated as follows:

$$\varepsilon_{ss}^0 = \frac{\partial u}{\partial s} - \frac{w}{R}$$

$$\kappa = \frac{\partial \varphi}{\partial s}$$

$$\gamma_{sz} = \frac{\partial u}{\partial s} - \varphi + \frac{w}{R} \quad (8)$$

Where the shear strain of bending axis is denoted by γ and the strain at any points is evaluated as:

$$\varepsilon_{ss} = \varepsilon_{ss}^0 + z \kappa \quad (9)$$

Hamilton principle is defined in $t=0$ and $t=T$ as follows:

$$\int_0^t (\delta U - \delta K + \delta V) dt = 0 \quad (10)$$

Where U shows the total strain energy, K denotes the Kinetic energy and W is the performed work. The first variation of the total strain energy is calculated as:

$$\delta U = \int_V \sigma_{ij} \delta \varepsilon_{ij} dV = \int_V (\sigma_{ss} \delta \varepsilon_{ss} + \sigma_{sz} \delta \gamma_{sz}) dV =$$

$$\int_0^L \left(N \left(\frac{\delta u}{R} + \frac{\partial \delta w}{\partial s} \right) + M \left(\frac{\partial \delta \varphi}{\partial s} \right) + Q \left(\frac{\partial \delta w}{\partial s} - \delta \varphi + \frac{\delta w}{R} \right) \right) ds \quad (11)$$

Where N, M and Q are the axial normal force, the bending moment and the shear force respectively which are defined as follows:

$$N = \int_A \sigma_{ss} dA, \quad M = \int_A \sigma_{ss} z dA, \quad Q = \int_A K_{shear} \sigma_{sz} dA \quad (12)$$

Where K_{shear} is the shear correction coefficient. The first variation of the performed work is evaluated as follows:

$$\delta V = - \int (f \delta u + p \delta w) ds \quad (13)$$

Where f and p denotes the radial and tangential distributed load, respectively. The first variation of kinetic energy is calculated as follows:

$$\delta K = \frac{1}{2} \int_0^L \delta \int_A \rho(z) \left(\left(\frac{\partial u_s}{\partial t} \right)^2 + \left(\frac{\partial u_z}{\partial t} \right)^2 \right) dA ds \delta T = \int_0^L \left(I_0 \left(\frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} \right) + \right.$$

$$\left. \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} + \left(I_1 + \frac{I_2}{R} \right) \left(\frac{\partial \varphi}{\partial t} \frac{\partial \delta w}{\partial t} + \frac{\partial w}{\partial t} \frac{\partial \delta \varphi}{\partial t} \right) + I_2 \frac{\partial \varphi}{\partial t} \frac{\partial \delta \varphi}{\partial t} \right) ds \quad (14)$$

Substituting Eqs. (11), (13) and (14) into Eq. (10) gives:

$$\int_0^T \left(\int_0^L \left(N \left(\frac{\delta u}{R} + \frac{\partial \delta w}{\partial s} \right) + M \left(\frac{\partial \delta \varphi}{\partial s} \right) + Q \left(\frac{\partial \delta w}{\partial s} - \delta \varphi + \frac{\delta w}{R} \right) \right) ds - \right.$$

$$\left. \int_0^L \left(I_0 \left(\frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} \right) + \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} + \left(I_1 + \frac{I_2}{R} \right) \left(\frac{\partial \varphi}{\partial t} \frac{\partial \delta w}{\partial t} + \frac{\partial w}{\partial t} \frac{\partial \delta \varphi}{\partial t} \right) + I_2 \frac{\partial \varphi}{\partial t} \frac{\partial \delta \varphi}{\partial t} \right) ds - \right.$$

$$\left. \int_0^L (f \delta u + p \delta w) ds \right) dt = 0 \quad (15)$$

Using part by part integration, differential equation and related boundary conditions are obtained:

$$\int_0^T \left(\int_0^L \left(\left(-\frac{N}{R} + \frac{\partial Q}{\partial s} - f + I_0 \frac{\partial^2 u}{\partial t^2} \right) \delta u + \left(-\frac{\partial N}{\partial s} + \frac{Q}{R} - p + \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) \frac{\partial^2 w}{\partial t^2} + \left(I_1 + \frac{I_2}{R} \right) \frac{\partial^2 \varphi}{\partial t^2} \right) \delta w + \right.$$

$$\left. \left(-\frac{\partial M}{\partial s} - Q + \left(I_1 + \frac{I_2}{R} \right) \frac{\partial^2 w}{\partial t^2} + I_2 \frac{\partial^2 \varphi}{\partial t^2} \right) \delta \varphi \right) ds \right) dt - \int_0^T [N \delta w + M \delta \varphi + Q \delta u]_0^L dt = 0 \quad (16)$$

Considering Eq. (16) the differential equation of motion is calculated as follows:

$$\frac{N}{R} + \frac{\partial Q}{\partial s} = I_0 \frac{\partial^2 u}{\partial t^2} \quad (17)$$

$$\frac{\partial N}{\partial s} - \frac{Q}{R} = \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) \frac{\partial^2 w}{\partial t^2} + \left(I_1 + \frac{I_2}{R} \right) \frac{\partial^2 \varphi}{\partial t^2} \quad (18)$$

$$\frac{\partial M}{\partial s} + Q = \left(I_1 + \frac{I_2}{R} \right) \frac{\partial^2 w}{\partial t^2} + I_2 \frac{\partial^2 \varphi}{\partial t^2} \quad (19)$$

Substituting nonlocal parameter into Eq. (12), axial normal force, bending moment and shear force leads to:

$$N - \mu^2 \frac{\partial^2 N}{\partial s^2} = A \left(-\frac{u}{R} + \frac{\partial w}{\partial s} \right) + B \frac{\partial \varphi}{\partial s} \quad (20)$$

$$M - \mu^2 \frac{\partial^2 M}{\partial s^2} = B \left(-\frac{u}{R} + \frac{\partial w}{\partial s} \right) + D \frac{\partial \varphi}{\partial s} \quad (21)$$

$$Q - \mu^2 \frac{\partial^2 Q}{\partial s^2} = K_{shear} C \left(\frac{w}{R} + \frac{\partial u}{\partial s} - \varphi \right) \quad (22)$$

$$(I_0, I_1, I_2) = \int_A \rho(z) (1, z, z^2) dA \quad (A, B, D) = \int_A E(z) (1, z, z^2) dA, \text{ and}$$

$$C = \int_A \frac{E(z)}{1 + \nu(z)} dA \quad (23)$$

Substituting Eqs.(17), (18) and (19) into nonlocal Eqs. (20), (21) and (22), the differential equation per displacement in the nonlocal theory for a curved FG nanobeam is obtained as follows:

$$A \left(-\frac{u}{R^2} + \frac{\partial w}{R \partial s} \right) + B \frac{\partial \varphi}{R \partial s} + K_{shear} C \left(\frac{\partial w}{R \partial s} + \frac{\partial^2 u}{\partial s^2} - \frac{\partial \varphi}{\partial s} \right) = I_0 \frac{\partial^2 u}{\partial t^2} - \mu^2 I_0 \frac{\partial^4 u}{\partial t^2 \partial s^2} \quad (24)$$

$$A \left(-\frac{\partial u}{R \partial s} + \frac{\partial^2 w}{\partial s^2} \right) + B \frac{\partial^2 \varphi}{\partial s^2} - K_{shear} C \left(\frac{w}{R^2} + \frac{\partial u}{R \partial s} - \frac{\varphi}{R} \right) =$$

$$\left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) \frac{\partial^2 w}{\partial t^2} - \mu^2 \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) \frac{\partial^4 w}{\partial s^2 \partial t^2} + \left(I_1 + \frac{I_2}{R} \right) \frac{\partial^2 \varphi}{\partial t^2} -$$

$$\mu^2 \left(I_1 + \frac{I_2}{R} \right) \frac{\partial^4 \varphi}{\partial s^2 \partial t^2} \quad (25)$$

$$B \left(-\frac{\partial u}{R \partial s} + \frac{\partial^2 w}{\partial s^2} \right) + D \frac{\partial^2 \varphi}{\partial s^2} + K_{shear} C \left(\frac{w}{R} + \frac{\partial u}{\partial s} - \varphi \right) =$$

$$\left(I_1 + \frac{I_2}{R} \right) \frac{\partial^2 w}{\partial t^2} - \mu^2 \left(I_1 + \frac{I_2}{R} \right) \frac{\partial^4 w}{\partial s^2 \partial t^2} + I_2 \frac{\partial^2 \varphi}{\partial t^2} - \mu^2 I_2 \frac{\partial^4 \varphi}{\partial s^2 \partial t^2} \quad (26)$$

5. Analytical Solution of Curved FG Nanobeam

An analytical solution of a curved FG nanobeam in simply supported boundary condition is proposed in this section. The displacement was chosen in a way that satisfies the boundary condition as follows:

$$w(s, t) = \sum_{n=1}^{\infty} W_n \cos\left(\frac{n\pi}{L}s\right) e^{i\omega_n t} \quad (27)$$

$$u(s, t) = \sum_{n=1}^{\infty} U_n \sin\left(\frac{n\pi}{L}s\right) e^{i\omega_n t} \quad (28)$$

$$\varphi(s, t) = \sum_{n=1}^{\infty} \varphi_n \cos\left(\frac{n\pi}{L}s\right) e^{i\omega_n t} \quad (29)$$

Where $i = \sqrt{-1}$, u_n, w_n and φ_n are Fourier coefficients that are obtained for different values of mode number.

6. Bending Analysis

Variables u_0, w_0 and φ are time independence in static analysis where $u_0 = u_0(s)$, $w_0 = w_0(s)$ and $\varphi = \varphi(s)$. As a consequence all the time derivatives in Eqs. (24), (25) and (26) are eliminated. It is assumed that only the radial distributed force is applied to the curved nanobeam which is expanded based in Fourier expansion that is calculated in the following equation:

$$f(s) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi}{L}s\right) \quad (30)$$

The Fourier coefficient is calculated as follows:

$$F_n = \frac{2}{L} \int_0^L f(s) \sin\left(\frac{n\pi}{L}s\right) ds \quad (31)$$

For different types of loading F_n is calculated as follows:

$$f(s) = f_0, \quad F_n = \frac{4f_0}{n\pi}, \quad n = 1, 3, 5, \dots \quad (32)$$

$$f(s) = \frac{f_0 s}{L}, \quad F_n = \frac{2f_0}{n\pi} (-1)^{n+1}, \quad n = 1, 2, 3, \dots \quad (33)$$

$$f(s) = f_0 \delta(s - s_0) \frac{s}{L}, \quad F_n = 2f_0 \sin\left(\frac{n\pi}{L}s_0\right), \quad n = 1, 2, 3, \dots \quad (34)$$

$$f(s) = f_0 \sin\left(\frac{\pi s}{L}\right), \quad F_1 = f_0, F_n = 0, \quad n = 2, 3, \dots \quad (35)$$

Substituting Eqs. (27)-(30) into Eqs. (24), (25) and (26) and eliminating the time dependent terms, the following matrix is obtained:

$$\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \begin{bmatrix} U_n \\ W_n \\ \varphi_n \end{bmatrix} = \begin{bmatrix} F_n \lambda_n \\ 0 \\ 0 \end{bmatrix} \quad (36)$$

Where the stiffness matrix and λ_n is calculated as follows:

$$\begin{aligned} k_{11} &= -\frac{A}{R^2} - Ck_{shear} \left(\frac{n\pi}{L}\right)^2, & k_{12} &= k_{21} = -\frac{A}{R} \left(\frac{n\pi}{L}\right) - \frac{Ck_{shear}}{R^2} \left(\frac{n\pi}{L}\right), \\ k_{13} &= k_{31} = -\frac{B}{R} \left(\frac{n\pi}{L}\right) + Ck_{shear} \left(\frac{n\pi}{L}\right), & k_{22} &= -A \left(\frac{n\pi}{L}\right)^2 - \frac{Ck_{shear}}{R^2}, \\ k_{23} &= k_{32} = -B \left(\frac{n\pi}{L}\right)^2 + \frac{Ck_{shear}}{R}, & k_{33} &= -D \left(\frac{n\pi}{L}\right)^2 - Ck_{shear}, \\ \lambda_n &= -\left(1 + \mu^2 \left(\frac{n\pi}{L}\right)^2\right) \end{aligned} \quad (37)$$

Solving the abovementioned equation, u_n, w_n and φ_n are obtained as:

$$U_n = \frac{F_n \lambda_n (k_{23}^2 - k_{22} k_{33})}{\xi} \quad (38)$$

$$W_n = \frac{F_n \lambda_n (-k_{13} k_{23} + k_{12} k_{33})}{\xi} \quad (39)$$

$$\psi_n = \frac{F_n \lambda_n (k_{13} k_{22} - k_{12} k_{33})}{\xi} \quad (40)$$

Where ξ is calculated as follows:

$$\xi = k_{13}^2 k_{22} - 2k_{12} k_{13} k_{23} + k_{12}^2 k_{33} + k_{11} (k_{23}^2 - k_{22} k_{33}) \quad (41)$$

7. Vibration Analysis

The free vibration of a curved FG nanobeam is analyzed in this section. The radial and tangential forces are not considered in this analysis. Substituting Eqs. (27)-(29) into Eqs. (24)-(26) leads to:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} + \omega_n^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} U_n \\ W_n \\ \varphi_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (42)$$

$$\begin{aligned} m_{11} &= I_0 + I_0 \mu^2 \left(\frac{n\pi}{L} \right)^2, & m_{12} &= m_{21} = 0, & m_{13} &= m_{31} = 0, \\ m_{22} &= I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} + \mu^2 \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) \left(\frac{n\pi}{L} \right)^2, \\ m_{23} &= m_{32} = I_1 + \frac{I_2}{R} + \mu^2 \left(I_1 + \frac{I_2}{R} \right) \left(\frac{n\pi}{L} \right)^2, & m_{33} &= I_2 + I_2 \mu^2 \left(\frac{n\pi}{L} \right)^2. \end{aligned} \quad (43)$$

In order to obtain the non-trivial solution for $U_n \neq 0$, $W_n \neq 0$ and $\psi_n \neq 0$, it is necessary to set the determinant of $[k] + \omega^2 [m] = 0$ equal to zero.

8. Results and Discussion

Several numerical examples to study the influence of the opening angle parameter, nonlocal parameter, thickness ratio, material distribution, and mode numbers are presented in this section. The material properties of Steel and Alumina (Al_2O_3) are utilized in this work. For steel $E=210$ GPa, $\rho=7800$ Kg/m³ and for Alumina $E=390$ GPa, $\rho=3960$ Kg/m³.

9. Validation

As mentioned, when the radius of curved FG nanobeam and its opening angle increases and decreases, its mechanical behavior approaches to those of straight one. For this purpose a comparison has been made between the

vibration behavior of a curved FG nanobeam with very slight opening angle of $\alpha = \frac{\pi}{360} = 0.5$ and a straight nanobeam^[24]. The dimensionless natural frequency $\left(\bar{\omega} = \omega \times L^2 \sqrt{\rho_c A / E_c I} \right)$ of a curved FG nanobeam with very slight opening angle and a straight nanobeam for $L = 10000 \times 10^{-9}$, $b = 10000 \times 10^{-9}$ and $k_s = \frac{5}{6}$ is compared in Table 1. It is concluded that our results are in a good agreement with Ref.^[24].

L/h	μ	P							
		0		0.5		1		10	
		Present	Rahmani et al ^[24]	Present	Rahmani et al ^[24]	Present	Rahmani et al ^[24]	Present	Rahmani et al ^[24]
20	0	9.82945	9.8296	7.71442	7.7149	6.96707	6.9676	5.65187	5.6521
	1×10^{-12}	9.37758	9.3777	7.35978	7.3602	6.64678	6.6473	5.39204	5.3923
	2×10^{-12}	8.98279	8.9829	7.04994	7.0504	6.36695	6.3674	5.16504	5.1653
	3×10^{-12}	8.63400	8.6341	6.77620	6.7766	6.11973	6.1202	4.96449	4.9647
	4×10^{-12}	8.32292	8.3230	6.53206	6.5325	5.89924	5.8997	4.78562	4.7858

	5×10^{-12}	8.04321	8.0433	6.31253	6.3129	5.70099	5.7014	4.62479	4.6250
50	0	9.86304	9.8631	7.74106	7.7413	6.99147	6.9917	5.67283	5.6730
	1×10^{-12}	9.40962	9.4097	7.38519	7.3854	6.67006	6.6703	5.41204	5.4122
	2×10^{-12}	9.01349	9.0136	7.07428	7.0745	6.38926	6.3895	5.18420	5.1843
	3×10^{-12}	8.66351	8.6636	6.79960	6.7998	6.14117	6.1414	4.98290	4.9830
	4×10^{-12}	8.35136	8.3515	6.55461	6.5548	5.91991	5.9201	4.80337	4.8035
	5×10^{-12}	8.07070	8.0708	6.33433	6.3345	5.72096	5.7212	4.64195	4.6421
100	0	9.86788	9.8680	7.74494	7.7451	6.99505	6.9952	5.67587	5.6760
	1×10^{-12}	9.41425	9.4143	7.38891	7.3891	6.67347	6.6736	5.41494	5.4150
	2×10^{-12}	9.01792	9.0180	7.07784	7.0780	6.39252	6.3927	5.18698	5.1871
	3×10^{-12}	8.66775	8.6678	6.80301	6.8032	6.14430	6.1444	4.98557	4.9857
	4×10^{-12}	8.35547	8.3555	6.55791	6.5580	5.92293	5.9231	4.80594	4.8060
	5×10^{-12}	8.07466	8.0747	6.33752	6.3376	5.72388	5.7240	4.64444	4.6445

Table 1. Dimensionless frequencies for first mode

9. Free Vibration of Curved FG Nanobeam

The vibration of a curved FG nanobeam per different aspect ratios of 10, 25 and 75, power law indexes of 0, 0.1, 1 and 10, and opening angles of $\frac{3\pi}{4}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{9}$ are analyzed and non-dimensionalized in this section. The non-dimensionalized natural frequencies for nonlocal parameter $\mu=0$ are represented in Table 1.

As it is clear, increasing both the power index law and the opening angle results in decreasing the amount of frequency. Increasing the thickness is also results in decreasing the frequency of curved FG nanobeam. The amount of frequency of a curved FG nanobeam for nonlocal parameters of 1, 2, 3 and 4 is represented in Tables 2, 3, 4 and 5 respectively. Comparing the aforementioned tables it is clear that as the nonlocal parameter increases, the amount of non-dimensionalized frequency increases.

α	L/h=10				L/h=25				L/h=75			
	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5
$\pi/9$	9.534 86	8.381 26	6.723 27	5.711 59	9.662 99	8.505 22	6.835 33	5.807 95	9.685 33	8.530 18	6.860 90	5.828 80
$\pi/6$	9.315 67	8.180 15	6.553 50	5.569 78	9.440 42	8.305 76	6.671 48	5.669 80	9.462 16	8.332 43	6.700 64	5.693 03
$\pi/3$	8.193 67	7.178 71	5.733 71	4.876 71	8.301 63	7.296 89	5.853 62	4.976 58	8.320 42	7.324 65	5.887 66	5.002 97
$\pi/2$	6.519 51	5.707 94	4.552 00	3.870 98	6.604 18	5.803 01	4.652 13	3.955 10	6.618 89	5.826 09	4.682 04	3.978 54
$2\pi/3$	4.491 93	3.936 02	3.138 79	2.666 54	4.550 75	3.999 93	3.206 51	2.725 08	4.560 94	4.015 05	3.226 58	2.741 45
$3\pi/4$	3.400 11	2.981 97	2.379 07	2.019 62	3.445 51	3.029 54	2.429 04	2.063 74	3.453 37	3.040 41	2.443 47	2.075 88

Table 2. The variation of dimensionless frequencies $\bar{\omega} = \omega \times L^2 \sqrt{(\rho_c A)/(E_c I)}$ versus opening angle, power index law and aspect ratio for simply supported curved beam ($\mu=0$)

α	L/h=10				L/h=25				L/h=75			
	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5
$\pi/9$	9.096 52	7.995 96	6.414 18	5.449 02	9.218 77	8.114 22	6.521 10	5.540 95	9.240 08	8.138 04	6.545 49	5.560 84
$\pi/6$	8.887 41	7.804 10	6.252 22	5.313 73	9.006 42	7.923 93	6.364 78	5.409 15	9.027 17	7.949 38	6.392 60	5.431 31
$\pi/3$	7.816 99	6.848 69	5.470 12	4.652 52	7.919 99	6.961 44	5.584 52	4.747 79	7.937 91	6.987 92	5.617 00	4.772 97
$\pi/2$	6.219 80	5.445 53	4.342 73	3.693 02	6.300 57	5.536 23	4.438 27	3.773 27	6.314 60	5.558 26	4.466 80	3.795 64
$2\pi/3$	4.285 43	3.755 07	2.994 49	2.543 95	4.341 55	3.816 04	3.059 1	2.599 81	4.351 27	3.830 47	3.078 24	2.615 42
$3\pi/4$	3.243 8	2.844 89	2.269 7	1.926 77	3.287 12	2.890 27	2.317 37	1.968 86	3.294 62	2.900 63	2.331 14	1.980 46

Table 3. The variation of dimensionless frequencies $\bar{\omega} = \omega \times L^2 \sqrt{(\rho_c A)/(E_c I)}$ versus opening angle, power index law and aspect ratio for simply supported curved beam ($\mu=1\text{nm}^2$)

α	L/h=10				L/h=25				L/h=75			
	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5
$\pi/9$	8.713 57	7.659 33	6.144 15	5.219 62	8.830 67	7.772 62	6.246 57	5.307 68	8.851 08	7.795 43	6.269 93	5.326 74
$\pi/6$	8.513 26	7.475 55	5.989 01	5.090 03	8.627 26	7.590 34	6.096 82	5.181 43	8.647 13	7.614 71	6.123 48	5.202 65
$\pi/3$	7.487 91	6.560 37	5.239 83	4.456 66	7.586 56	6.668 37	5.349 42	4.547 92	7.603 74	6.693 73	5.380 52	4.572 03
$\pi/2$	5.957 95	5.216 28	4.159 91	3.537 55	6.035 32	5.303 16	4.251 42	3.614 42	6.048 76	5.324 26	4.278 75	3.635 85
$2\pi/3$	4.105 02	3.596 99	2.868 43	2.436 85	4.158 77	3.655 39	2.930 32	2.490 36	4.168 09	3.669 21	2.948 65	2.505 31
$3\pi/4$	3.107 24	2.725 12	2.174 15	1.845 66	3.148 73	2.768 59	2.219 81	1.885 98	3.155 91	2.778 52	2.233 00	1.897 08

Table 4. The variation of dimensionless frequencies $\bar{\omega} = \omega \times L^2 \sqrt{(\rho_c A)/(E_c I)}$ versus opening angle, power index law and aspect ratio for simply supported curved beam ($\mu=2\text{nm}^2$)

α	L/h=10				L/h=25				L/h=75			
	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5
$\pi/9$	8.375 23	7.361 93	5.905 59	5.016 95	8.487 79	7.470 82	6.004 02	5.101 59	8.507 41	7.492 75	6.026 48	5.119 91
$\pi/6$	8.182 70	7.185 29	5.756 47	4.892 39	8.292 28	7.295 62	5.860 09	4.980 24	8.311 38	7.319 05	5.885 71	5.000 64
$\pi/3$	7.197 16	6.305 64	5.036 38	4.283 61	7.291 99	6.409 45	5.141 71	4.371 33	7.308 50	6.433 83	5.171 61	4.394 51
$\pi/2$	5.726 61	5.013 74	3.998 39	3.400 19	5.800 98	5.097 25	4.086 34	3.474 08	5.813 90	5.117 53	4.112 61	3.494 67
$2\pi/3$	3.945 63	3.457 32	2.757 05	2.342 24	3.997 29	3.513 46	2.816 54	2.393 66	4.006 25	3.526 75	2.834 16	2.408 04
$3\pi/4$	2.986 59	2.619 31	2.089 73	1.773 99	3.026 47	2.661 09	2.133 62	1.812 75	3.033 38	2.670 63	2.146 30	1.823 42

Table 5. The variation of dimensionless frequencies $\bar{\omega} = \omega \times L^2 \sqrt{(\rho_c A)/(E_c I)}$ versus opening angle, power index law and aspect ratio for simply supported curved beam ($\mu=3\text{nm}^2$)

α	L/h=10				L/h=25				L/h=75			
	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5
$\pi/9$	8.073 48	7.096 69	5.692 81	4.836 19	8.181 97	7.201 65	5.787 70	4.917 78	8.200 89	7.222 79	5.809 35	4.935 44
$\pi/6$	7.887 88	6.926 41	5.549 07	4.716 12	7.993 51	7.032 76	5.648 96	4.800 81	8.011 92	7.055 35	5.673 65	4.820 47
$\pi/3$	6.937 85	6.078 45	4.854 92	4.129 27	7.029 26	6.178 52	4.956 45	4.213 83	7.045 18	6.202 02	4.985 28	4.236 18
$\pi/2$	5.520 28	4.833 10	3.854 33	3.277 69	5.591 98	4.913 60	3.939 11	3.348 91	5.604 43	4.933 14	3.964 44	3.368 77
$2\pi/3$	3.803 47	3.332 76	2.657 71	2.257 85	3.853 27	3.386 87	2.715 06	2.307 42	3.861 90	3.399 68	2.732 05	2.321 28
$3\pi/4$	2.878 98	2.524 93	2.014 44	1.710 08	2.917 43	2.565 21	2.056 75	1.747 43	2.924 09	2.574 41	2.068 97	1.757 72

Table 6. The variation of dimensionless frequencies $\bar{\omega} = \omega \times L^2 \sqrt{(\rho_c A)/(E_c I)}$ versus opening angle, power index law and aspect ratio for simply supported curved beam ($\mu=4\text{nm}^2$)

The non-dimensionalized frequency variation of a curved FG nanobeam versus the power index law for different L/h values, nonlocal parameter of $\mu=1$ and different opening angles of $\frac{3\pi}{4}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{9}$ is shown in Fig. 2a-2f. As seen, increasing the power index law decrease the amount of frequency which is more intense for $p < 2$. Also as l/h ratio increases the difference between the frequencies decreases. This leads to a fact that the shear effect decreases with increasing the length of nanotube. As observed in Tables 1-5 it is also concluded that as the opening angle increases the amount of frequency decreases.

Figure 2. The variation of non-dimensionalized frequencies $\bar{\omega} = \omega \times L^2 \sqrt{(\rho_c A)/(E_c I)}$ versus power law index for opening angles of $\frac{\pi}{18}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$ and $\frac{2\pi}{3}$ and nonlocal parameters of 0(a), 1(b), 2(c) and 3(d) (nm²).

The variation of non-dimensionalized frequency per different aspect ratios versus mode number and for nonlocal parameter of $\mu = 0, 1, 2, 3$ is shown in Fig.3a-3d respectively. The opening angle is assumed to be $\frac{\pi}{3}$. As predicted increasing the mode number tends to increase the frequency values. The importance of nonlocal parameter and non-classical theories is clarified by comparing Fig.3a with Fig.3b as the frequency of the sixth mode of Fig.3a is twice those of Fig.3b.

Figure 3. The variation of non-dimensionalized frequencies $\bar{\omega} = \omega \times L^2 \sqrt{(\rho_c A)/(E_c I)}$ versus first six vibration modes for

different nonlocal parameters of 0, 1, 2, 3 and 4(nm²), the opening angles of $\frac{\pi}{9}$ (a), $\frac{\pi}{2}$ (b), $\frac{2\pi}{3}$ (c) and $\frac{3\pi}{4}$ (d).(p=1)

10. Bending Analysis of Curved FG Nanobeam

Bending behavior of a curved FG nanobeam, which is under the distributed radial load, is analyzed in this section. The

non-dimensionalized displacements are calculated as $\bar{u}_n = \frac{100E_c I}{f_0 R^4 \alpha^4} u_n \left(\frac{\alpha}{2}\right)$, $\bar{w}_n = \frac{100E_c I}{f_0 R^4 \alpha^4} w_n(0)$ where $\bar{u}_n, \bar{w}_n$ denote the non-dimensionalized radial and tangential displacement, respectively. The non-dimensionalized displacement for nonlocal parameters of 0, 1, 2, 3 and 4 is represented in tables (7)-(11). As it can be seen the amount of displacement

increases as the opening angle increases which is more intense when the opening angle increases from $\frac{2\pi}{3}$ to $\frac{3\pi}{4}$. It is clear that increasing both the nonlocal parameter and the aspect ratio results in increasing and decreasing the non-dimensionalized radial displacement, respectively.

	α	L/h=10				L/h=25				L/h=75			
		P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5
Radial deflection	$\frac{\pi}{9}$	1.3668 9	1.5173 2	1.8375 6	2.0962 4	1.3400 6	1.4869 6	1.7983 6	2.0473 2	1.3355 2	1.4814 9	1.7906 9	2.0379 9
	$\frac{\pi}{6}$	1.4109 7	1.5672 7	1.9000 4	2.1671 5	1.3831 2	1.5351 3	1.8574 3	2.1144 1	1.3784 9	1.5291 9	1.8486 2	2.1038 6
	$\frac{\pi}{3}$	1.6901 3	1.8809 4	2.2875 2	2.6078 3	1.6557 1	1.8391 3	2.2281 3	2.5358 7	1.6498 8	1.8308 6	2.2142 5	2.5198 5
	$\frac{\pi}{2}$	2.379 2	2.6526 2	3.2361 1	3.6876 4	2.3281 1	2.5880 6	3.1395 9	3.5723 9	2.3194 9	2.5746 9	3.1150 9	3.5446 8
	$\frac{2\pi}{3}$	4.3474 5	4.8565 6	5.9432 6	6.7699 6	4.2483 2	4.7263 6	5.7408 9	6.5310 9	4.2315 4	4.6981 7	5.6869 3	6.4707 4
	$\frac{3\pi}{4}$	7.0207 8	7.8502 5	9.6216 3	10.957 9	6.8548 2	7.6291 3	9.2725 6	10.548 10.548	6.8267 3	7.5805 5	9.1778 9	10.442 5
Tangential deflection	$\frac{\pi}{9}$	0.2542 37	0.2723 32	0.3098 31	0.3542 38	0.2500 89	0.2732 36	0.3214 53	0.3647 07	0.2493 86	0.2748 87	0.3282 02	0.3713 38
	$\frac{\pi}{6}$	0.3935 4	0.4260 97	0.4940 99	0.5630 72	0.3871 21	0.4247 7	0.5033 91	0.5704 38	0.3860 34	0.4261 14	0.5099 7	0.5767 72
	$\frac{\pi}{3}$	0.9413 49	1.0297 1.0297	1.2153 2	1.3808 3	0.9260 22	1.0202 7	1.2175 2	1.3781 2	0.9234 27	1.0207 1.0207	1.2243 5	1.3842 2
	$\frac{\pi}{2}$	1.9826 9	2.1753 1	2.5805 2	2.9294 5	1.9505 1.9505	2.1516 3	2.5728 1	2.9112 1	1.9450 5	2.1508 1	2.5816 7	2.9184 5
	$\frac{2\pi}{3}$	4.8158 7	5.2901 1	6.2882 8	7.1361 1	4.7379 3	5.2290 4	6.2577 1	7.0798 3	4.7247 3	5.2253 9	6.2738 6	7.0919 8
	$\frac{3\pi}{4}$	8.7344 4	9.5974 7	11.414 2	12.952 12.952	8.5933 8.5933	9.4852 2	11.353 5	12.844 6	8.5694 1	9.4778 6	11.380 4	12.864 2

Table 7. The variation of dimensionless radial and tangential displacement versus opening angle, aspect ratios and power law index for simply supported curved beam ($\mu=0$)

	α	L/h=10				L/h=25				L/h=75			
		P=0	P=0, 2	P=1	P=5	P=0	P=0, 2	P=1	P=5	P=0	P=0, 2	P=1	P=5
Radial deflection	$\pi/9$	1.4976 8	1.6625 2	2.0133 9	2.2967 4	1.4687	1.6297	1.971	2.2438 4	1.4638	1.6237 9	1.9626 9	2.2337 4
	$\pi/6$	1.5460 9	1.7173 7	2.0820 1	2.3746 1	1.5159 9	1.6826 2	2.0358 7	2.3175 3	1.5109	1.6761 8	2.0263 1	2.3060 9
	$\pi/3$	1.8527 1	2.0618 9	2.5075 9	2.8586 2	1.8154 1	2.0165 3	2.4430 5	2.7804 5	1.8091	2.0075 4	2.4279 3	2.7629 6
	$\pi/2$	2.6094 1	2.9095 4	3.5495 5	4.0447 3	2.5540 4	2.8392 2	3.4441 8	3.9190 6	2.5446 7	2.8245 4	3.4175	3.8888
	$2\pi/3$	4.7719 3	5.3307 3	6.5235 4	7.4307 8	4.6635 7	5.1883 4	6.3019 4	7.1694 5	4.6452 3	5.1574 9	6.2429	7.1033 4
	$3\pi/4$	7.7089 3	8.6197 2	10.564 7	12.031 9	7.5272	8.3774 6	10.182 1	11.582 6	7.4964 3	8.3242	10.078 2	11.466 9
Tangential deflection	$\pi/9$	0.2797 48	0.2994 9	0.3403 9	0.3892 6	0.2751 06	0.3004 99	0.3533 92	0.4009 71	0.2743 2	0.3023 47	0.3609 43	0.4083 9
	$\pi/6$	0.4330 11	0.4686 63	0.5431 27	0.6190 32	0.4258 31	0.4671 75	0.5535 1	0.6272 62	0.4246 15	0.4686 78	0.5608 65	0.6343 42
	$\pi/3$	1.0355 3	1.1325 6	1.3363 9	1.5185	1.0184 4	1.1220 3	1.3388 1	1.5154 4	1.0155 4	1.1225	1.3464 1	1.5222 3
	$\pi/2$	2.1803 4	2.3919 8	2.8372 4	3.2210 2	2.1445 8	2.3656 5	2.8285 9	3.2006 7	2.1385 3	2.3647 3	2.8384	3.2086 8
	$2\pi/3$	5.2938 9	5.8150 2	6.9119 3	7.844	5.2077 3	5.7474 7	6.8779 8	7.7816 3	5.1931 4	5.7434 1	6.8957 7	7.795
	$3\pi/4$	9.5996 2	10.548	12.544 3	14.234 5	9.4439 4	10.424 1	12.477 1	14.115 9	9.4175 9	10.415 9	12.506 7	14.137 4

Table 8. The variation of dimensionless radial and tangential displacement versus opening angle, aspect ratios and power law index for simply supported curved beam ($\mu=1$)

	α	L/h=10				L/h=25				L/h=75			
		P=0	P=0, 2	P=1	P=5	P=0	P=0, 2	P=1	P=5	P=0	P=0, 2	P=1	P=5
Radial deflection	$\pi/9$	1.6284 6	1.8077 1	2.1892 2	2.4972 3	1.5973 4	1.7724 5	2.1436 4	2.4403 7	1.5920 8	1.7660 9	2.1346 9	2.4294 9
	$\pi/6$	1.6812 1	1.8674 7	2.2639 8	2.5820 7	1.6488 7	1.8301	2.2143 2	2.5206 5	1.6433 9	1.8231 8	2.2040 1	2.5083 2
	$\pi/3$	2.0152 9	2.2428 4	2.7276 5	3.1094 1	1.9751 1	2.1939 3	2.6579 7	3.0250 4	1.9683 1	2.1842 2	2.6416 1	3.0061 2
	$\pi/2$	2.8398 2	3.1664 7	3.863	4.4018 1	2.7799 7	3.0903 8	3.7488 6	4.2657 3	2.7698 4	3.0744 8	3.7199 1	4.2329 1
	$2\pi/3$	5.1963 5	5.8049 1	7.1038 2	8.0916 7	5.0788 2	5.6503 2	6.8630 7	7.8078 1	5.0589 2	5.6168	6.7988 8	7.7359 4
	$3\pi/4$	8.3970 7	9.3891 8	11.507 8	13.105 9	8.1995 7	9.1257 8	11.091 6	12.617 3	8.1661 3	9.0678 5	10.978 6	12.491 3
Tangential deflection	$\pi/9$	0.3052 58	0.3266 47	0.3709 49	0.4242 82	0.3001 22	0.3277 63	0.3853 31	0.4372 36	0.2992 53	0.3298 07	0.3936 83	0.4454 42
	$\pi/6$	0.4724 81	0.5112 29	0.5921 54	0.6749 93	0.4645 41	0.5095 81	0.6036 29	0.6840 85	0.4631 96	0.5112 42	0.6117 59	0.6919 13
	$\pi/3$	1.1297 2	1.2354 1	1.4574 6	1.6561 8	1.1108 5	1.2237 8	1.4601	1.6527 6	1.1076 6	1.2242 9	1.4684 7	1.6602 4
	$\pi/2$	2.3779 9	2.6086 6	3.0939 6	3.5125 9	2.3386 6	2.5796 8	3.0843 7	3.4901 3	2.332	2.5786 5	3.0951 3	3.4989 1
	$2\pi/3$	5.7719 2	6.3399 4	7.5355 8	8.5519	5.6775 2	6.2658 9	7.4982 4	8.4834 3	5.6615 4	6.2614 2	7.5176 8	8.4980 1
	$3\pi/4$	10.464 8	11.498 4	13.674 4	15.517	10.294 6	11.362 9	13.600 8	15.387 2	10.265 8	11.354	13.633	15.410 6

Table 9. The variation of dimensionless radial and tangential displacement versus opening angle, aspect ratios and power law index for simply supported curved beam ($\mu=2$)

	α	L/h=10				L/h=25				L/h=75			
		P=0	P=0. 2	P=1	P=5	P=0	P=0. 2	P=1	P=5	P=0	P=0. 2	P=1	P=5
Radial deflection	$\pi/9$	1.7592 5	1.9529	2.3650 6	2.6977 3	1.7259 9	1.9151 9	2.3162 8	2.6368 9	1.7203 5	1.9083 9	2.3066 9	2.6252 4
	$\pi/6$	1.8163 4	2.0175 7	2.4459 5	2.7895 3	1.7817 5	1.9775 8	2.3927 7	2.7237 7	1.7758 9	1.9701 7	2.3817 1	2.7105 5
	$\pi/3$	2.1778 7	2.4237 9	2.9477 1	3.3601 9	2.1348 2	2.3713 2	2.8728 8	3.2696 3	2.1275 3	2.3609 9	2.8552 9	3.2492 9
	$\pi/2$	3.0702 4	3.4233 9	4.1764 4	4.7588 9	3.0059 1	3.3415 4	4.0535 3	4.6124 3	2.9950 2	3.3244 2	4.0223 2	4.5770 2
	$2\pi/3$	5.6208 8	6.2790 8	7.6841 7	8.7525 6	5.4940 7	6.1122 9	7.4242 1	8.4461 7	5.4726 1	6.0761 1	7.3548 6	8.3685 5
	$3\pi/4$	9.0852 2	10.158 6	12.450 9	14.179 9	8.8719 4	9.8741 1	12.001 1	13.651 9	8.8358 3	9.8115 9	11.878 9	13.515 7
Tangential deflection	$\pi/9$	0.3307 69	0.3538 05	0.4015 08	0.4593 04	0.3251 39	0.3550 27	0.4172 69	0.4735 86	0.3241 86	0.3572 67	0.4264 23	0.4824 94
	$\pi/6$	0.5119 51	0.5537 94	0.6411 82	0.7309 53	0.5032 5	0.5519 87	0.6537 48	0.7409 09	0.5017 77	0.5538 06	0.6626 54	0.7494 83
	$\pi/3$	1.2239 6	1.3382 3	1.5785 3	1.7938 5	1.2032 7	1.3255 3	1.5813 9	1.7900 8	1.1997 8	1.3260 9	1.5905 4	1.7982 5
	$\pi/2$	2.5756 4	2.8253 3	3.3506 8	3.8041 6	2.5327 4	2.7937 6	3.3401 6	3.7796 8	2.5254 8	2.7925 7	3.3518 5	3.7891 3
	$2\pi/3$	6.2499 4	6.8648 5	8.1592 3	9.2598 2	6.1473 2	6.7843 1	8.1185 1	9.1852 3	6.1299 5	6.7794 4	8.1395 9	9.2010 3
	$3\pi/4$	11.33 9	12.448 5	14.804 5	16.799 6	11.145 2	12.301 8	14.724 4	16.658 4	11.113 9	12.292 1	14.759 3	16.683 8

Table 10. The variation of dimensionless radial and tangential displacement versus opening angle, aspect ratios and power law index for simply supported curved beam ($\mu=3$)

	α	L/h=10				L/h=25				L/h=75			
		P=0	P=0. 2	P=1	P=5	P=0	P=0. 2	P=1	P=5	P=0	P=0. 2	P=1	P=5
Radial deflection	$\pi/9$	1.8900 4	2.0980 9	2.5408 9	2.8982 3	1.8546 3	2.0579 4	2.4889 1	2.8334 2	1.8486 3	2.0506 9	2.4786 9	2.8209 9
	$\pi/6$	1.9514 6	2.1676 7	2.6279 2	2.9969 9	1.9146 2	2.1250 7	2.5712 1	2.9269 9	1.9083 9	2.1171 6	2.5594 9	2.9127 8
	$\pi/3$	2.3404 5	2.6047 4	3.1677 8	3.6109 8	2.2945 2	2.5487 2	3.0878 1	3.5142 1	2.2867 4	2.5375 8	3.0689 7	3.4924 5
	$\pi/2$	3.3006 5	3.6803 2	4.4898 9	5.1159 7	3.2318 4	3.5927 1	4.3582 1	4.9590 7	3.2201 9	3.5743 6	4.3247 3	4.9211 3
	$2\pi/3$	6.0452 5	6.7532 5	8.2643 8	9.4134 4	5.9093 2	6.5742 7	7.9853 4	9.0845 4	5.8863 9	6.5354 2	7.9108 3	9.0011 5
	$3\pi/4$	9.7733 6	10.928 1	13.394 8	15.253 8	9.5443 1	10.622 4	12.910 7	14.686 5	9.5055 3	10.555 2	12.779 3	14.540 1
Tangential deflection	$\pi/9$	0.3562 8	0.3809 62	0.4320 67	0.4943 27	0.3501 56	0.3822 91	0.4492 08	0.5097 64	0.3491 19	0.3847 27	0.4591 64	0.5195 46
	$\pi/6$	0.5514 21	0.5963 6	0.6902 1	0.7869 14	0.5419 6	0.5943 93	0.7038 68	0.7977 33	0.5403 58	0.5963 69	0.7135 49	0.8070 53
	$\pi/3$	1.3180 9	1.4411 2	1.6996 2	1.9315 2	1.2956 9	1.4272 8	1.7026 8	1.9274 9	1.2918 9	1.4278 9	1.7126 5	1.9362 5
	$\pi/2$	2.7732 9	3.0420 1	3.6074 3	4.0957 3	2.7268 2	3.0077 2	3.5959 4	4.0690 6	2.7189 6	3.0064 9	3.6085 8	4.0793 6
	$2\pi/3$	6.7279 7	7.3897 7	8.7828 8	9.9676 9	6.6171 2	7.3027 3	8.7387 8	9.8870 4	6.5983 5	7.2974 5	8.7614 9	9.9040 4
	$3\pi/4$	12.195 2	13.399 4	15.934 6	18.082 1	11.995 9	13.240 6	15.848 7	17.929 7	11.962 1	13.230 1	15.885 6	17.957 8

Table 11. The variation of dimensionless radial and tangential displacement versus opening angle, aspect ratios and power law index for simply supported curved beam ($\mu=4$)

The variation of radial displacement versus the power law index for different aspect ratios and for opening angles of

$\frac{3\pi}{4}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{9}$ is shown in Fig. 4(a-f). The difference between the radial displacement for low aspect ratios and the radial displacement in high aspect ratios is considerable which increases as the power law index increases. It should be noted that in the vibration analysis increasing the aspect ratio results in increasing the frequencies while in bending analysis increasing the mentioned values tends to decrease the radial displacement.

Figure4. The variation of dimensionless radial displacement versus power law index and for selected aspect ratios and for opening angle simply supported curved beam ($\mu=4$)

11. Conclusion

In this work, the nonlocal Timoshenko beam model of a curved beam is used to analyze the bending and vibration problem. Using the Hamilton principle and the nonlocal theory the differential equations and governing equations for a curved FG nanobeam were obtained respectively. The Navier method was employed to propose a vibration and bending solution. The influence of some parameters such as curved FG nanobeam opening angle, nonlocal parameter, mode number, aspect ratio and the influence of power index law of FGM on the non-dimensionalized frequency, non-dimensionalized radial displacement and non-dimensionalized tangential displacement were thoroughly studied in this work by employing several tables and graphs. The results revealed that increasing the aspect ratio results in increasing and decreasing the non-dimensionalized frequency and displacements respectively. As the nonlocal parameter increases the amount of frequency and the radial and tangential displacement decreases and increases respectively. It was also concluded that the non-dimensionalized frequency and the radial and tangential displacement decreases and increases with increasing the opening angle respectively which shows the importance of the angle parameter.

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